

# Rational curves with many rational points over a finite field

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## Abstract

We study a particular plane curve over a finite field whose normalization is of genus 0. The number of rational points of this curve achieves the Aubry-Perret bound for rational curves. The configuration of its rational points and a generalization of the curve are also presented.

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## 1 Introduction

Let  $C$  be a curve of degree  $d$  in projective plane  $\mathbb{P}^2$  over a finite field  $\mathbb{F}_q$ . We are interested in the number  $N_q(C)$  of the set  $C(\mathbb{F}_q)$  of  $\mathbb{F}_q$ -points. Especially we want to give a good upper bound for  $N_q(C)$  in terms of  $d$  and  $q$  for curves

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$C$  in a certain class. In 2010, the second and the third authors proved a fact of this direction [9].

**Theorem** *If a plane curve  $C$  of degree  $d \geq 2$  over  $\mathbb{F}_q$  has no  $\mathbb{F}_q$ -linear components, then*

$$N_q(C) \leq (d-1)q + 1 \quad (1)$$

*unless  $d = q = 4$  and  $C$  is projectively equivalent to*

$$K : (X + Y + Z)^4 + (XY + YZ + ZX)^2 + XYZ(X + Y + Z) = 0$$

*over  $\mathbb{F}_4$ . In this exceptional case,  $N_4(K) = 14$ .*

Since the upper bound (1) was originally conjectured by Sziklai [15], we refer it as the Sziklai bound. Note that the bound (1) makes sense only if  $2 \leq d \leq q + 2$  because it is worse than the obvious bound:

$$N_q(C) \leq \#\mathbb{P}^2(\mathbb{F}_q) = q^2 + q + 1$$

if  $d > q + 2$ .

The study of curves that attain the Sziklai bound is still under way [8, 10], but we have not yet met any example of a curve with singularities which attains it. We guess that there will be a better bound for curves with singularities, which is our motivation for focusing on them.

Let  $C'$  be an irreducible curve of degree  $d$  in  $\mathbb{P}^2$  over  $\mathbb{F}_q$  whose normalization is  $\mathbb{P}^1$ . Since the morphism  $\mathbb{P}^1 \rightarrow C'$  given by the normalization is defined over  $\mathbb{F}_q$ , the number of nonsingular  $\mathbb{F}_q$ -points of  $C'$  is at most  $q + 1$ . Therefore

$$N_q(C') \leq q + 1 + \frac{1}{2}(d-1)(d-2), \quad (2)$$

because the number of singularities of  $C'$  is at most  $\frac{1}{2}(d-1)(d-2)$ . This bound is a special case of a result of Aubry and Perret [1, Prop. 2.3]. So we refer this bound as the Aubry-Perret bound for rational curves. The bound (2) is, of course, better than (1) in the meaningful range of  $d$ .

Let  $B$  be the rational plane curve over  $\mathbb{F}_q$  defined by the image of

$$\mathbb{P}^1 \ni (s, t) \rightarrow (s^{q+1}, s^q t + s t^q, t^{q+1}) \in \mathbb{P}^2. \quad (3)$$

$B$  is of degree  $q + 1$  and  $N_q(B)$  actually attains the Aubry-Perret bound for  $d = q + 1$ . This curve over an algebraically closed field containing  $\mathbb{F}_q$  appeared in Ballico and Hefez's classification list [2, Th. 1] of non-reflexive plane curves of degree  $q + 1$  with second-order  $q$  in a different parametrization

$$(s, t) \rightarrow (t^{q+1}, s(s-t)^q, s^q(s-t)) \in \mathbb{P}^2$$

from (3). Actually those parametrizations are equivalent over  $\mathbb{F}_{q^2}$ , but not over  $\mathbb{F}_q$ . However we refer the curve  $B$  parametrized by (3) as the Ballico-Hefez curve. Recently, in [3] the first author has studied the Galois points of  $B$  by using a parametrization

$$(s, t) \rightarrow (s^{q+1}, (s+t)^{q+1}, t^{q+1}) \in \mathbb{P}^2$$

equivalent to (3) over  $\mathbb{F}_q$ , and has found that the constellation of Galois points of  $B$  is described in a similar way that the second author did in [7] for Hermitian curves.

As Hermitian curves have many interesting properties in finite geometry including coding theory, Ballico-Hefez curves also might have lovely properties because of this similarity<sup>1</sup>.

In Section 2, we verify that the number of  $\mathbb{F}_q$ -points of  $B$  is actually  $q + 1 + \frac{1}{2}q(q-1)$ , in other words  $B$  is a singular maximal curve, and show that the zeta function of  $B$  is given by  $\frac{(1+T)^{\frac{q^2-q}{2}}}{(1-T)(1-qT)}$ . In Section 3, we consider the case  $q$  is odd. We give combinatorial characterizations of the set of  $\mathbb{F}_q$ -points of  $B$ , and compute the  $\tau_i$ 's, where  $\tau_i$  is the number of  $\mathbb{F}_q$ -lines that are  $i$ -secants to  $B(\mathbb{F}_q)$ . In Section 4, we compute the  $\tau_i$ 's for  $q$  even. By using those results, we compute parameters of codes coming from  $B(\mathbb{F}_q)$  in Section 5. Some of them have the largest minimum distance under fixed length and dimension. In the last section, we propose a generalization of the Ballico-Hefez curve, which is a rational curve in  $\mathbb{P}^n$  parametrized by elementary symmetric polynomials in  $t, t^q, \dots, t^{q^{n-1}}$ . We give a formula of the number of  $\mathbb{F}_q$ -points of this curve.

## 2 Arithmetic of the curve $B$

We study the arithmetic properties of the Ballico-Hefez curve  $B$ . We prepare some notations.  $\Phi$  denotes the morphism (3), that is,  $\Phi(s, t) = (s^{q+1}, s^q t + s t^q, t^{q+1})$ , and  $\varphi$  denotes  $\Phi|_{\mathbb{A}^1}$ , where  $\mathbb{A}^1 = \{s \neq 0\}$ , that is,  $\varphi(t) = (1, t + t^q, t^{q+1})$ . For a point  $P = (\alpha, \beta) \in \mathbb{P}^1$ ,  $P^q$  denotes the point  $(\alpha^q, \beta^q) \in \mathbb{P}^1$ .  $\text{Sing } B$  denotes the set of singularities of  $B$ , and  $\text{Flex } B$  the set of inflection points of  $B$ . The precise definition of an “inflection point” in our case will be given in 2.1 below. We also take interest in the order-sequence and the  $q$ -Frobenius order-sequence for  $\Phi$ . The notion of  $q$ -Frobenius order-sequence was introduced by Stöhr and Voloch [14] as a tool to bound the number of  $\mathbb{F}_q$ -points of curves. Here we give the definition of them in a little more general setting than our original one.

**2.1 Order-sequence and  $q$ -Frobenius order-sequence.** Let  $f : \tilde{C} \rightarrow \mathbb{P}^n$  be a morphism from a nonsingular curve  $\tilde{C}$  over  $\overline{\mathbb{F}}_q$ , where  $\overline{\mathbb{F}}_q$  denotes the

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<sup>1</sup> For other similarity, see Remark 2.5.

algebraic closure of  $\mathbb{F}_q$ . In a neighborhood of an assigned point  $P \in \tilde{C}$ ,  $f$  can be represented by  $(n+1)$ -tuple regular functions  $(f_0, \dots, f_n)$  one of which is the constant function 1. Let  $t$  be a local parameter at  $P$ . Then regular functions around  $P$  can be embedded into the formal power series ring  $\overline{\mathbb{F}}_q[[t]]$  via the identification with the completion of the local ring at  $P \in \tilde{C}$ . The  $i$ -th Hasse derivation  $D^{(i)}$  on  $\overline{\mathbb{F}}_q[[t]]$  is given by  $D^{(i)}t^k = \binom{k}{i}t^{k-i}$ . Then  $\{(\varepsilon_0, \dots, \varepsilon_n) \mid \varepsilon_0 < \dots < \varepsilon_n, \det((D^{(\varepsilon_i)}f_j)(P))_{i,j} \neq 0\}$  is nonempty. The minimum  $(n+1)$ -tuple in the above set by the lexicographic order is called the Hermitian  $P$ -invariant for  $f$ . The Hermitian  $P$ -invariant is constant if  $P$  is in a certain nonempty open subset of  $\tilde{C}$ . The Hermitian  $P$ -invariant of a point in this open subset is called the order-sequence for  $f$ . For details, consult [6, Ch.7].

Let  $C = f(\tilde{C}) \subset \mathbb{P}^n$ , and  $\varepsilon_0 < \dots < \varepsilon_n$  the order-sequence for  $f$ . For a nonsingular point  $P' = f(P) \in C$ , the linear subspace spanned by  $\nu + 1$  vectors  $\{((D^{(\varepsilon_i)}f_0)(P), \dots, (D^{(\varepsilon_i)}f_n)(P)) \mid i = 0, 1, \dots, \nu\}$  in  $\mathbb{P}^n$  is the tangent  $\nu$ -plane at  $P'$ , which is denoted by  $T_{P'}^{(\nu)}C$ .

If the image of  $P'$  by  $q$ -Frobenius map lies on  $T_{P'}^{(\nu)}C$  with some  $\nu < n$  for almost all  $P' \in C$ , the curve  $C$  is said to be  $q$ -Frobenius nonclassical, and the minimum number  $\nu$  having the above property is called the  $q$ -Frobenius index of  $C$  (see, [4, Prop. 2]). Let  $\nu$  be the  $q$ -Frobenius index of  $C$ . The sequence  $\{\varepsilon_0, \varepsilon_1, \dots, \varepsilon_n\} \setminus \{\varepsilon_\nu\}$  is called the  $q$ -Frobenius order-sequence of  $C$ .

For a plane curve  $C = f(\tilde{C}) \subset \mathbb{P}^2$ , a nonsingular point  $P' \in C$  is an inflection point if  $i(C, T_{P'}^{(1)}(C); P') > \varepsilon_2$ .

Now we go back to our original setting.

**Theorem 2.2** (i) *For any  $P \in \mathbb{P}^1$ , the map induced by  $\Phi$  on tangent spaces*

$$d\Phi_P : T_{P, \mathbb{P}^1} \rightarrow T_{\Phi(P), \mathbb{P}^2}$$

*is injective.*

(ii) *Sing  $B$  consists of  $\frac{q^2-q}{2}$  ordinary double points.*

(iii) *For  $P \in \mathbb{P}^1$ ,  $\Phi(P) \in \text{Sing } B$  if and only if  $P \in \mathbb{P}^1(\mathbb{F}_{q^2}) \setminus \mathbb{P}^1(\mathbb{F}_q)$ . In this case,  $\Phi^{-1}(\Phi(P)) = \{P, P^q\}$ .*

(iv) *The order-sequence for  $\Phi$  is  $\{0, 1, q\}$ , and  $B$  is  $q$ -Frobenius nonclassical.*

(v)  *$\Phi(P)$  is an inflection point of  $B$  if and only if  $P \in \mathbb{P}^1(\mathbb{F}_q)$ .*

(vi)  *$B(\mathbb{F}_q) = \text{Flex } B \cup \text{Sing } B$ .*

*Proof.* The assertion (i) is obvious because

$$\begin{pmatrix} \frac{\partial \Phi}{\partial s} \\ \frac{\partial \Phi}{\partial t} \end{pmatrix} = \begin{pmatrix} s^q & t^q & 0 \\ 0 & s^q & t^q \end{pmatrix}. \quad (4)$$

By (i),  $\Phi(P)$  is a singular point if and only if  $\#\Phi^{-1}(\Phi(P)) > 1$ . We want to determine such points. For  $P_\infty = (0, 1)$ , obviously  $\Phi^{-1}(\Phi(P_\infty)) = \{P_\infty\}$ . For two distinct points  $P_\alpha = (1, \alpha)$  and  $P_\beta = (1, \beta) \in \mathbb{A}^1$ ,  $\Phi(P_\alpha) = \Phi(P_\beta)$  if and only if  $\varphi(\alpha) = \varphi(\beta)$ , that is,

$$\begin{cases} \alpha + \alpha^q &= \beta + \beta^q \\ \alpha^{q+1} &= \beta^{q+1}. \end{cases}$$

Hence  $\Phi(P_\alpha) = \Phi(P_\beta)$  implies that  $(X - \alpha)(X - \alpha^q) = (X - \beta)(X - \beta^q)$ . Since  $\alpha \neq \beta$ , we have  $\alpha = \beta^q$  and  $\alpha^q = \beta$ . Hence  $P_\alpha, P_\beta \in \mathbb{P}^1(\mathbb{F}_{q^2}) \setminus \mathbb{P}^1(\mathbb{F}_q)$  with  $P_\alpha^q = P_\beta$ . Conversely, this condition obviously leads to  $\Phi(P_\alpha) = \Phi(P_\beta)$ . Therefore  $\Phi^{-1}(\text{Sing } B) = \mathbb{P}^1(\mathbb{F}_{q^2}) \setminus \mathbb{P}^1(\mathbb{F}_q)$  and  $\mathbb{P}^1(\mathbb{F}_{q^2}) \setminus \mathbb{P}^1(\mathbb{F}_q) \xrightarrow{\Phi} \text{Sing } B$  is a 2-to-1 map. Moreover, since the tangent line to  $B$  at the branch corresponding  $P_\alpha \in \mathbb{P}^1$  is

$$\begin{vmatrix} X & Y & Z \\ 1 & \alpha^q & 0 \\ 0 & 1 & \alpha^q \end{vmatrix} = 0, \quad (5)$$

$\Phi(P_\alpha) = \Phi(P_{\alpha^q}) \in \text{Sing } B$  is an ordinary double point, that is, those two branches have different tangent lines. So (ii) and (iii) have been established, and furthermore we have also established

$$B(\mathbb{F}_q) = \Phi(\mathbb{P}^1(\mathbb{F}_q)) \cup \text{Sing } B \quad (6)$$

because  $\alpha + \alpha^q$  and  $\alpha^{q+1}$  are the trace and the norm from  $\mathbb{F}_{q^2}$  to  $\mathbb{F}_q$  respectively.

Next we calculate the order-sequence for  $\Phi$ . Since  $d\Phi \neq 0$ , the first two orders are 0 and 1. Let  $D_t^{(\nu)}$  be the  $\nu$ -th Hasse derivation on the function field  $\overline{\mathbb{F}}_q(B) = \overline{\mathbb{F}}_q(t)$  with respect to  $t$ . Since

$$\begin{aligned} \det \begin{pmatrix} \varphi \\ D_t^{(1)} \varphi \\ D_t^{(\nu)} \varphi \end{pmatrix} &= \det \begin{pmatrix} 1 & t + t^q & t^{q+1} \\ 0 & 1 & t^q \\ 0 & \binom{q}{\nu} t^{q-\nu} & \binom{q+1}{\nu} t^{q+1-\nu} \end{pmatrix} \\ &= \begin{cases} 0 & (q > \nu > 1) \\ t - t^q & (\nu = q) \end{cases}, \end{aligned}$$

the order-sequence for  $\Phi$  is  $\{0, 1, q\}$ .

Since the tangent line at  $\Phi(P_\alpha)$  is given by (5),  $P_\alpha^q = (1, (\alpha^q + \alpha)^q, (\alpha^{q+1})^q)$  lies on it. Hence  $B$  is  $q$ -Frobenius nonclassical. The set of inflection points of  $B$  is given by the support of the Wronskian divisor:

$$(1 + 2)(q + 1)P_\infty + \text{div} \left( \det \begin{pmatrix} \varphi \\ D_t^{(1)} \varphi \\ D_t^{(q)} \varphi \end{pmatrix} \right) + (0 + 1 + q)\text{div } dt = \sum_{P \in \mathbb{P}^1(\mathbb{F}_q)} P.$$

Hence  $\text{Flex } B = \Phi(\mathbb{P}^1(\mathbb{F}_q))$ , which, together with (6), implies (vi).  $\square$

As was mentioned in Introduction,  $N_q(B)$  attains the Aubry-Perret bound for rational curves.

**Corollary 2.3** *The Ballico-Hefez curve  $B$  is of degree  $q+1$ , and  $N_q(B) = q+1 + \frac{q(q-1)}{2}$ .*

**Corollary 2.4** *The zeta function of  $B$  is  $Z_B(T) = \frac{(1+T)^{\frac{q^2-q}{2}}}{(1-T)(1-qT)}$ .*

*Proof.* Since  $\Phi : \mathbb{P}^1 \rightarrow B$  is the normalization of  $B$ , we can calculate the zeta function of  $B$  by [1, Th. 2.1] together with the following informations:  $\text{Sing } B$  consists of  $\frac{q^2-q}{2}$  points that are  $\mathbb{F}_q$ -rational, and  $\Phi^{-1}(\Phi(P)) = \{P, P^q\}$  with  $P \in \mathbb{P}^1(\mathbb{F}_{q^2}) \setminus \mathbb{P}^1(\mathbb{F}_q)$ .  $\square$

**Remark 2.5** Corollaries 2.3 and 2.4 suggest another similarity between the Ballico-Hefez curve and the Hermitian curve. In general, if  $C'$  is an irreducible curve over  $\mathbb{F}_q$  with the normalization  $\mathbb{P}^1 \xrightarrow{\pi} C'$ , then the zeta function of  $C'$  is of the form  $\frac{L_{C'}(T)}{(1-T)(1-qT)}$  where  $L_{C'}(T)$  is a polynomial of degree  $\#\pi^{-1}(\text{Sing } C') - \#\text{Sing } C'$  (say,  $\Delta_{C'}$ ). Let  $\{\beta_1, \dots, \beta_{\Delta_{C'}}\}$  be the set of reciprocal roots of  $L_{C'}$ . Then  $|\beta_i| = 1$  ( $i = 1, \dots, \Delta_{C'}$ ) and  $N_{q^r}(C') = q^r + 1 - \sum_{i=1}^{\Delta_{C'}} \beta_i^r$  (see, [1]). Especially,  $N_q(C') \leq q+1 + \Delta_{C'}$ , and if equality holds, then we have  $\Delta_{C'} \leq \frac{q^2-q}{2}$  by Ihara's argument [11]. Actually equality holds in both inequalities for the Ballico-Hefez curve. Comparing with Rück and Stichtenoth's characterization of Hermitian curves [12], we might expect Ballico-Hefez curves to be characterized among rational curves by those two properties.

The following lemma will be used later.

**Lemma 2.6** *For each  $\mathbb{F}_q$ -point  $Q$  of  $\mathbb{P}^2$  which does not lie on  $B$ , there are two points  $P_1, P_2 \in B(\mathbb{F}_q) \setminus \text{Sing } B$  such that  $T_{P_1}B \cap T_{P_2}B = \{Q\}$ . Moreover the pair  $\{P_1, P_2\}$  is uniquely determined by  $Q$ .*

*Proof.* Since  $i(B, T_{P_i}B; P_i) = q+1$  by Theorem 2.2,  $T_{P_i}B \cap B = \{P_i\}$ . Hence the map

$$\begin{aligned} S^2(B(\mathbb{F}_q) \setminus \text{Sing } B) \setminus \Delta &\rightarrow \mathbb{P}^2(\mathbb{F}_q) \setminus B(\mathbb{F}_q) \\ \{P_1, P_2\} &\mapsto T_{P_1}B \cap T_{P_2}B \end{aligned}$$

is well-defined, where  $S^2(B(\mathbb{F}_q) \setminus \text{Sing } B)$  denotes the symmetric product of  $B(\mathbb{F}_q) \setminus \text{Sing } B$  and  $\Delta$  its diagonal subset. Since the source and the target of this map have the same cardinality  $\frac{q(q+1)}{2}$ , it is enough to show the following fact; Let  $P'_i = (\alpha_i, \beta_i) \in \mathbb{P}^1$  ( $i = 1, 2, 3$ ) be three distinct points. Then three embedded tangent lines  $d\Phi(T_{P'_i, \mathbb{P}^1})$  ( $i = 1, 2, 3$ ) are not concurrent. In fact,

since  $d\Phi(T_{P'_i, \mathbb{P}^1})$  is spanned by two vectors  $(\alpha_i^q, \beta_i^q, 0)$  and  $(0, \alpha_i^q, \beta_i^q)$  (see (4)), its equation is  $\beta_i^{2q}X - \alpha_i^q\beta_i^qY + \alpha_i^{2q}Z = 0$ . Since

$$\begin{vmatrix} \beta_1^{2q} & -\alpha_1^q\beta_1^q & \alpha_1^{2q} \\ \beta_2^{2q} & -\alpha_2^q\beta_2^q & \alpha_2^{2q} \\ \beta_3^{2q} & -\alpha_3^q\beta_3^q & \alpha_3^{2q} \end{vmatrix} = -\prod_{i<j}(\alpha_i\beta_j - \alpha_j\beta_i)^q,$$

those three lines are not concurrent.  $\square$

### 3 Geometry of $B$ with $\mathbb{F}_q$ -lines, for $q$ odd

The projective plane of  $\mathbb{F}_q$ -lines in  $\mathbb{P}^2$  is denoted by  $\check{\mathbb{P}}^2(\mathbb{F}_q)$ . An  $\mathbb{F}_q$ -line  $l$  is an  $i$ -line if  $\#(B(\mathbb{F}_q) \cap l) = i$ . The cardinality of  $\{l \in \check{\mathbb{P}}^2(\mathbb{F}_q) | l \text{ is an } i\text{-line}\}$  is denoted by  $\tau_i$ . Since  $\deg B = q + 1$ , only  $q + 2$  numbers  $\{\tau_i | 0 \leq i \leq q + 1\}$  make sense. The purpose of this and next sections is to determine the exact values of the  $\tau_i$ 's.

In this section, we assume  $q$  is odd.

**Lemma 3.1** *Let  $C_B$  be the conic defined by  $4XZ - Y^2 = 0$ . Then  $B \cap C_B = B(\mathbb{F}_q) \cap C_B(\mathbb{F}_q) = B(\mathbb{F}_q) \setminus \text{Sing } B$ , and  $T_P(B) = T_P(C_B)$  for any  $P \in B \cap C_B$ .*

*Proof.* Let  $\Phi(\alpha, \beta) = P \in B(\mathbb{F}_q) \setminus \text{Sing } B$ . Then  $(\alpha, \beta) \in \mathbb{P}^1(\mathbb{F}_q)$ . Hence we may suppose  $\alpha, \beta \in \mathbb{F}_q$ , and have  $P = (\alpha^{q+1}, \alpha^q\beta + \alpha\beta^q, \beta^{q+1}) = (\alpha^2, 2\alpha\beta, \beta^2)$ , which lies on  $C_B$ . So

$$B(\mathbb{F}_q) \setminus \text{Sing } B \subseteq C_B(\mathbb{F}_q) \cap B(\mathbb{F}_q) \subseteq B \cap C_B.$$

Let  $\mathbf{b}_1(P) = (\alpha^q, \beta^q, 0)$ ,  $\mathbf{b}_2(P) = (0, \alpha^q, \beta^q)$ ,  $\mathbf{c}_1(P) = (2\alpha, 2\beta, 0)$ , and  $\mathbf{c}_2(P) = (0, 2\alpha, 2\beta)$ . Then  $T_PB$  is spanned by  $\mathbf{b}_1(P)$  and  $\mathbf{b}_2(P)$ , and  $T_PC_B$  by  $\mathbf{c}_1(P)$  and  $\mathbf{c}_2(P)$ . Since  $\alpha, \beta \in \mathbb{F}_q$ ,  $T_PB = T_PC_B$ . Hence

$$2(q+1) = (B.C_B) = \sum_{P \in B(\mathbb{F}_q) \setminus \text{Sing } B} i(B.C_B; P) \geq 2(q+1),$$

which means  $B \cap C_B = B(\mathbb{F}_q) \setminus \text{Sing } B$ .  $\square$

Now we give geometric characterization of the set of rational points  $B(\mathbb{F}_q)$  of the Ballico-Hefez curve for odd  $q$ . The configuration of  $B(\mathbb{F}_q)$  is more or less known. We define two subsets  $\mathcal{S}$  and  $\mathcal{T}$  in  $\mathbb{P}^2(\mathbb{F}_q)$ .

**Definition 3.2** (1) Let  $l_1, \dots, l_{q+1}$  be  $q+1$   $\mathbb{F}_q$ -lines that form an arc in  $\check{\mathbb{P}}^2(\mathbb{F}_q)$ , that is, no three of the  $q+1$  lines are concurrent.  $\mathcal{S}$  is the set  $\mathbb{P}^2(\mathbb{F}_q) \setminus \bigcup_{i<j} l_i \cap l_j$ .

(2) Let  $D$  be a conic<sup>2</sup> over  $\mathbb{F}_q$ .  $\mathcal{T}$  denotes the internal points of  $D(\mathbb{F}_q)$  together with  $D(\mathbb{F}_q)$ , which appeared in [5, Example 12.6 (3)].

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<sup>2</sup>In our context, the word ‘conic’ connotes that it is absolutely irreducible.

**Theorem 3.3** *The three subsets  $B(\mathbb{F}_q)$ ,  $\mathcal{S}$  and  $\mathcal{T}$  in  $\mathbb{P}^2(\mathbb{F}_q)$  are projectively equivalent over  $\mathbb{F}_q$ .*

*Proof.* First we show that  $\mathcal{S}$  can be constructed by the same way of constructing  $\mathcal{T}$ . By a theorem of Segre [13],  $l_1, \dots, l_{q+1}$  lie on a conic  $D'$  in  $\check{\mathbb{P}}^2(\mathbb{F}_q)$ . Then the dual  $D$  of  $D'$  in  $\mathbb{P}^2(\mathbb{F}_q)$  is also a conic, and each  $l_i$  tangents to  $D$ . Hence the point set  $\{l_i \cap l_j | 1 \leq i < j \leq q+1\}$  is the external points of  $D(\mathbb{F}_q)$ .

Next we consider the case for  $B(\mathbb{F}_q)$  and  $\mathcal{T}$ . By Lemmas 2.6 and 3.1,  $\mathbb{P}^2(\mathbb{F}_q) \setminus B(\mathbb{F}_q)$  is the external points of  $C_B(\mathbb{F}_q)$ . Therefore  $B(\mathbb{F}_q)$  coincides with  $\mathcal{T}$  made from  $C_B$ . Since any two conics over  $\mathbb{F}_q$  are projectively equivalent, so are those three sets.  $\square$

**Corollary 3.4** (i) *If  $\tau_i \neq 0$ , then  $i = 1$  or  $\frac{q+1}{2}$  or  $\frac{q+3}{2}$ .*

(ii)  $\tau_1 = q+1$ ;  $\tau_{\frac{q+1}{2}} = \frac{q(q-1)}{2}$ ;  $\tau_{\frac{q+3}{2}} = \frac{q(q+1)}{2}$ .

*Proof.* We count these numbers by using the configuration  $\mathcal{T}$  in Definition 3.2. Let  $l$  be an  $\mathbb{F}_q$ -line which tangents to the conic  $D$  at  $P \in D(\mathbb{F}_q)$ . For each point  $Q \in l(\mathbb{F}_q) \setminus \{P\}$ , there exists another tangent  $\mathbb{F}_q$ -line passing through  $Q$ , that is,  $Q$  is an external point of  $D(\mathbb{F}_q)$ . Therefore any tangent line is 1-line. For an  $\mathbb{F}_q$ -line which does not tangent to  $D$  at any  $\mathbb{F}_q$ -points, there are exactly  $\frac{q+1}{2}$  external points if the line does not meet  $D(\mathbb{F}_q)$ , or  $\frac{q-1}{2}$  if it meets  $D(\mathbb{F}_q)$ , because there are exactly two  $\mathbb{F}_q$ -tangent lines passing through an external point of  $D(\mathbb{F}_q)$ . Hence such a line is a  $\frac{q+1}{2}$ -line or a  $\frac{q-1}{2}$ -line.

For (ii),  $\tau_1 = \#D(\mathbb{F}_q) = q+1$ ;  $\tau_{\frac{q+3}{2}}$  is the same as the number of  $\mathbb{F}_q$ -lines joining two distinct points of  $D(\mathbb{F}_q)$ , which is  $\frac{q(q+1)}{2}$ . Hence  $\tau_{\frac{q+1}{2}} = q^2 + q + 1 - \tau_1 - \tau_{\frac{q+3}{2}} = \frac{q(q-1)}{2}$ .  $\square$

**Remark 3.5** The proof of Corollary 3.4 tells us a characterization of  $i$ -lines ( $i = 1, \frac{q+1}{2}, \frac{q+3}{2}$ ) for  $\mathcal{T}$ . Let  $l$  be an  $\mathbb{F}_q$ -line of  $\mathbb{P}^2$ . Then  $l$  is a 1-line for  $\mathcal{T}$  if and only if  $\#(l \cap D(\mathbb{F}_q)) = 1$ ; a  $\frac{q+1}{2}$ -line for  $\mathcal{T}$  if and only if  $\#(l \cap D(\mathbb{F}_q)) = 0$ ; a  $\frac{q-1}{2}$ -line for  $\mathcal{T}$  if and only if  $\#(l \cap D(\mathbb{F}_q)) = 2$ .

## 4 Geometry of $B$ with $\mathbb{F}_q$ -lines, for $q$ even

The aim of this section is to determine the  $\tau_i$ 's for  $B(\mathbb{F}_q)$  with  $q$  even, which corresponds to Corollary 3.4 for odd  $q$ .

**Proposition 4.1** *Suppose  $q$  is a power of 2.*

(i) *The  $q+1$  points of  $B(\mathbb{F}_q) \setminus \text{Sing } B$  are collinear.*



(ii) If  $\tau_i \neq 0$ , then  $i = 1$  or  $\frac{q}{2} + 1$  or  $q + 1$ .

(iii)  $\tau_1 = q + 1$ ;  $\tau_{\frac{q}{2}+1} = q^2 - 1$ ;  $\tau_{q+1} = 1$ .

*Proof.* Let  $\Phi(\alpha, \beta) = P \in B(\mathbb{F}_q) \setminus \text{Sing } B$ . Since  $(\alpha, \beta) \in \mathbb{P}^1(\mathbb{F}_q)$ ,  $P = (\alpha^2, 0, \beta^2)$ , that is,  $P$  lies on the line  $Y = 0$ . Let  $l_0 = \{Y = 0\}$ . Obviously  $l_0(\mathbb{F}_q) = B(\mathbb{F}_q) \setminus \text{Sing } B$  and  $l_0$  is a  $(q + 1)$ -line. For any  $\mathbb{F}_q$ -line  $l \neq l_0$ , put  $n(l) = \#(l \cap \text{Sing } B)$ . Since any tangent line to a branch at a singular point  $P$  of  $B$  is not  $\mathbb{F}_q$ -line,  $i(B.l; P) = 2$  for any  $\mathbb{F}_q$ -line  $l$  passing through  $P$ . Hence  $1 + 2n(l) \leq q + 1$ . Fix a point  $P_0 \in l_0(\mathbb{F}_q)$ , and count the number of points of  $\text{Sing } B$  by using  $\mathbb{F}_q$ -lines passing through  $P_0$ . Note that  $l_0$  and  $T_{P_0}B$  never meet  $\text{Sing } B$ . So we have

$$\frac{q^2 - q}{2} = \sum_{\substack{l \ni P_0 \\ \text{with} \\ l \neq l_0, T_{P_0}B}} n(l) \leq \frac{q}{2}(q - 1).$$

Therefore an  $\mathbb{F}_q$ -line  $l$  which neither  $l_0$  nor the tangent line to  $B$  at an  $\mathbb{F}_q$ -point of  $l_0$  is a  $(\frac{q}{2} + 1)$ -line, and the number of such lines is  $(q + 1)(q - 1)$ . Obviously the tangent line to  $B$  at an  $\mathbb{F}_q$ -point of  $l_0$  is a 1-line, and the number of such lines is  $q + 1$ . This completes the proof.  $\square$

## 5 Codes from Ballico-Hefez curves

**5.1 Codes from a subset of  $\mathbb{P}^2(\mathbb{F}_q)$ .** Let  $S$  be a subset of  $\mathbb{P}^2(\mathbb{F}_q)$  which consists of  $s$  elements. For each point  $P \in S$ , we fix a representative  $(a_0, a_1, a_2)$  of its coordinates with  $a_0, a_1, a_2 \in \mathbb{F}_q$ . Then for any homogeneous polynomial  $F(X_0, X_1, X_2)$  over  $\mathbb{F}_q$ , the value  $F(P) \in \mathbb{F}_q$  is determined without ambiguity.

Let  $\Gamma(\mathcal{O}(i))$  be the vector space of homogeneous polynomials over  $\mathbb{F}_q$  of degree  $i$ . Then the image of  $\mathbb{F}_q$ -linear map

$$\Gamma(\mathcal{O}(i)) \ni F \mapsto (F(P))_{P \in S} \in (\mathbb{F}_q)^s$$

gives a linear codes, which is denoted by  $C_L(S, \mathcal{O}(i))$ .

**Proposition 5.2** *Suppose that  $q$  is odd.*

- (i)  $C_L(B(\mathbb{F}_q), \mathcal{O}(1))$  is a  $\left[\frac{q^2+q+2}{2}, 3, \frac{q^2-1}{2}\right]_q$ -code, and it achieves the Griesmer bound.
- (ii) For  $q \geq 5$ ,  $C_L(B(\mathbb{F}_q), \mathcal{O}(2))$  is a  $\left[\frac{q^2+q+2}{2}, 6, \frac{q^2-q-4}{2}\right]_q$ -code.
- (iii) For  $q \geq 7$ ,  $C_L(B(\mathbb{F}_q), \mathcal{O}(3))$  is a  $\left[\frac{q^2+q+2}{2}, 10, \frac{q^2-2q-7}{2}\right]_q$ -code.

*Proof.* (i) The parameters are known by Corollaries 2.3 and 3.4. Since  $\frac{q^2-1}{2} = \frac{q-1}{2} \cdot q + \frac{q-1}{2}$ ,

$$\sum_{i=0}^2 \left\lceil \left( \frac{q^2-1}{2} \right) / q^i \right\rceil = \frac{q^2-1}{2} + \left( \frac{q-1}{2} + 1 \right) + 1 = \frac{q^2+q+2}{2},$$

which means the triple of parameters  $\left[ \frac{q^2+q+2}{2}, 3, \frac{q^2-1}{2} \right]_q$  achieves the Griesmer bound.

(ii) Let  $D$  be a curve over  $\mathbb{F}_q$  of degree 2 in  $\mathbb{P}^2$ . If  $D$  is absolutely irreducible, then  $\#D(\mathbb{F}_q) = q+1$ ; if it is irreducible over  $\mathbb{F}_q$  but not absolutely, then  $\#D(\mathbb{F}_q) = 1$ . Hence  $\#(B(\mathbb{F}_q) \cap D) \leq q+1$  for those two cases. If  $D$  is reducible over  $\mathbb{F}_q$ , then  $\#(B(\mathbb{F}_q) \cap D) \leq q+3$  by Corollary 3.4. In particular, no degree-two-curve contains  $B(\mathbb{F}_q)$ , and  $C_L(B(\mathbb{F}_q), \mathcal{O}(2))$  is a  $\left[ \frac{q^2+q+2}{2}, 6, \geq \frac{q^2-q-4}{2} \right]_q$ -code, where  $\geq \frac{q^2-q-4}{2}$  at the third parameter means the minimum distance of this code is at least  $\frac{q^2-q-4}{2}$ . For an external point  $Q$  of  $C_B(\mathbb{F}_q)$ , which does not lie on  $B(\mathbb{F}_q)$  by Theorem 3.3, there are exactly two  $\mathbb{F}_q$ -lines passing through  $Q$  each of which tangents to  $C_B$ . Hence there are exactly  $\frac{q-1}{2}$   $\mathbb{F}_q$ -lines passing through  $Q$  each of which meets  $C_B(\mathbb{F}_q)$  at two points. Since  $q \geq 5$ , we can choose two lines from those  $\frac{q-1}{2}$  lines, which are  $\frac{q+3}{2}$ -lines by Remark 3.5. The union of those two lines gives a codeword of weight  $\frac{q^2-q-4}{2}$ .

(iii) For  $C_L(B(\mathbb{F}_q), \mathcal{O}(3))$ , in order to verify that its parameters are  $\left[ \frac{q^2+q+2}{2}, 10, \geq \frac{q^2-2q-7}{2} \right]_q$ , it is enough to see that

$$\#(B(\mathbb{F}_q) \cap D) \leq \frac{3}{2}(q+3) \quad (7)$$

for any curve  $D$  over  $\mathbb{F}_q$  of degree 3. If  $D$  is absolutely irreducible, then  $\#(B(\mathbb{F}_q) \cap D) \leq q+1+2\sqrt{q} \leq \frac{3}{2}(q+3)$ . In other cases, one can verify (7) easily by using Corollary 3.4. Equality in (7) holds if one takes three internal lines of  $C_B(\mathbb{F}_q)$  passing through an assigned external point, which is possible because  $q \geq 7$ .  $\square$

When  $q$  is a power of 2,  $B(\mathbb{F}_q)$  contains  $q+1$  collinear points, so we can't expect to obtain good codes from the total set.

**Proposition 5.3** *Let  $q = 2^e$  with  $e > 2$ . Then  $C_L(\text{Sing } B, \mathcal{O}(1))$  is a  $\left[ \frac{q^2-q}{2}, 3, \frac{q^2-2q}{2} \right]_q$ -code, which achieves the Griesmer bound.*

*Proof.* From Proposition 4.1 and its proof, an  $i$ -line for  $\text{Sing } B$  exists if and only if  $i = 0$  or  $\frac{q}{2}$ . Hence we have the first half of the assertion. Since  $\frac{q^2-2q}{2} = \frac{q-2}{2} \cdot q$ , we have the additional assertion.  $\square$

## 6 Generalization of the curve $B$

We generalize the Ballico-Hefez curve in  $\mathbb{P}^2$  to rational curves in higher dimensional projective spaces. We will discuss only on the number of rational points on them.

**Notation 6.1** Let  $X_0, \dots, X_{n-1}$  be variables. We consider  $n+1$  elementary symmetric polynomials:

$$\sigma_k(X_0, \dots, X_{n-1}) := \sum_{i_1 < \dots < i_k} X_{i_1} \cdots X_{i_k} \quad (k = 0, \dots, n),$$

where we understand  $\sigma_0 = 1$ . We also consider “homogeneous” elementary symmetric polynomials of degree  $n$ :

$$\begin{aligned} \tilde{\sigma}_k(X_0, \dots, X_{n-1}; Y_0, \dots, Y_{n-1}) := \\ \sum_{\substack{i_1 < \dots < i_k; j_1 < \dots < j_{n-k} \\ \text{with} \\ \{i_1, \dots, i_k, j_1, \dots, j_{n-k}\} = \{0, \dots, n-1\}}} X_{i_1} \cdots X_{i_k} Y_{j_1} \cdots Y_{j_{n-k}}. \end{aligned}$$

**Definition 6.2** We define  $n+1$  homogeneous polynomials of degree  $q^{n-1} + q^{n-2} + \dots + 1$  in  $s$  and  $t$  by

$$\tilde{\varphi}_k(s, t) := \tilde{\sigma}_k(s, s^q, \dots, s^{q^{n-1}}; t, t^q, \dots, t^{q^{n-1}}),$$

and inhomogeneous ones by

$$\varphi_k(s) := \tilde{\varphi}_k(s, 1) = \sigma_k(s, s^q, \dots, s^{q^{n-1}}) = \sum_{i_1 < \dots < i_k} s^{q^{i_1}} \cdots s^{q^{i_k}}$$

for  $k = 0, 1, \dots, n$ .

$B_n$  denotes a curve in  $\mathbb{P}^n$  over  $\mathbb{F}_q$  defined by the image of

$$\Phi_n : \mathbb{P}^1 \ni (s, t) \mapsto (\tilde{\varphi}_0(s, t), \tilde{\varphi}_1(s, t), \dots, \tilde{\varphi}_n(s, t)) \in \mathbb{P}^n.$$

Since  $B_2 = B$ , the curve  $B_n$  is a generalization of the Ballico-Hefez curve.

**Remark 6.3**  $B_n$  is nondegenerate in  $\mathbb{P}^n$ , that is, there is no hyperplane of  $\mathbb{P}^n$  containing  $B_n$ .

In fact, suppose  $\sum_{i=0}^n \alpha_i \varphi_i(s) = 0$  for  $\alpha_0, \dots, \alpha_n \in \overline{\mathbb{F}}_q$ . Let  $D_s^{(\nu)}$  be the  $\nu$ -th Hasse derivation with respect to  $s$ . Note that for  $k$  and  $i_0 < \dots < i_l$ ,  $D_s^{(1+q+q^2+\dots+q^k)} s^{q^{i_0}+q^{i_1}+\dots+q^{i_l}} \neq 0$  if and only if  $l \geq k$  and  $i_\mu = \mu$  ( $\mu = 0, \dots, l$ ), and in this case,  $D_s^{(1+q+q^2+\dots+q^k)} s^{q^{i_0}+q^{i_1}+\dots+q^{i_l}} = s^{q^{i_k+1}+\dots+q^{i_l}}$ . Therefore

$$0 = D_s^{(1+q+q^2+\dots+q^k)} \left( \sum_{i=0}^n \alpha_i \varphi_i(s) \right) \Big|_{s=0} = \alpha_k,$$

which means that no hyperplane contains  $B_n$ .

**Theorem 6.4** (i) For  $P \in \mathbb{P}^1$ , the map arising from  $\Phi_n$  on tangent spaces

$$d\Phi_{n,P} : T_{P,\mathbb{P}^1} \rightarrow T_{\Phi_n(P),\mathbb{P}^n}$$

is injective.

(ii) For  $P \in \mathbb{P}^1$ ,  $\#\Phi_n^{-1}(\Phi_n(P)) > 1$  if and only if  $P \in \mathbb{P}^1(\mathbb{F}_{q^n}) \setminus \mathbb{P}^1(\mathbb{F}_q)$ . In this case, if  $\mathbb{F}_q(P) = \mathbb{F}_{q^k}$ , then

$$\Phi_n^{-1}(\Phi_n(P)) = \{P, P^q, \dots, P^{q^{k-1}}\},$$

and

$$\Phi_n(P) = \Phi_n(P^q) = \dots = \Phi_n(P^{q^{k-1}}) \in \mathbb{P}^n(\mathbb{F}_q).$$

(iii)  $B_n(\mathbb{F}_q) = \Phi_n(\mathbb{P}^1(\mathbb{F}_{q^n}))$ .

*Proof.* (i) Since

$$\begin{pmatrix} \frac{\partial \Phi_n}{\partial s} \\ \frac{\partial \Phi_n}{\partial t} \end{pmatrix} = \begin{pmatrix} 0 & t^{q+\dots+q^{n-1}} & * & \dots & * & * & s^{q+\dots+q^{n-1}} \\ t^{q+\dots+q^{n-1}} & * & * & \dots & * & s^{q+\dots+q^{n-1}} & 0 \end{pmatrix},$$

the rank of  $\begin{pmatrix} \frac{\partial \Phi_n}{\partial s} \\ \frac{\partial \Phi_n}{\partial t} \end{pmatrix}$  is 2 for any point  $(s, t) \in \mathbb{P}^1$ .

(ii) We may suppose  $t = 1$ . For two points  $P_\alpha = (\alpha, 1), P_\beta = (\beta, 1) \in \mathbb{P}^1$ ,  $\Phi_n(P_\alpha) = \Phi_n(P_\beta)$  if and only if  $\varphi_k(\alpha) = \varphi_k(\beta)$  for  $k = 0, 1, \dots, n$ . Since  $\prod_{i=0}^{n-1} (X - s^{q^i}) = \sum_{k=0}^n (-1)^k \varphi_k(s) X^{n-k}$ , those conditions are equivalent to the condition

$$\{\alpha, \alpha^q, \dots, \alpha^{q^{n-1}}\} = \{\beta, \beta^q, \dots, \beta^{q^{n-1}}\}$$

with counting multiplicity. Here we need the following lemma, which completes the proof of (ii).

**Lemma 6.5** Let two elements  $\alpha$  and  $\beta$  of  $\overline{\mathbb{F}_q}$  be distinct from each other. If  $\{\alpha, \alpha^q, \dots, \alpha^{q^{n-1}}\} = \{\beta, \beta^q, \dots, \beta^{q^{n-1}}\}$  with counting multiplicity, then  $\mathbb{F}_q(\alpha) = \mathbb{F}_q(\beta)$  is a subfield of  $\mathbb{F}_{q^n}$ .

*Proof.* Let  $\mathbb{F}_q(\alpha) = \mathbb{F}_{q^k}$ . Obviously  $\mathbb{F}_q(\beta) = \mathbb{F}_{q^k}$  also. Since  $\beta = \alpha^{q^i}$  for some  $i$  with  $1 \leq i \leq n-1$ ,  $\alpha^{q^n} = (\alpha^{q^i})^{q^{n-i}} = \beta^{q^{n-i}} \in \{\alpha, \alpha^q, \dots, \alpha^{q^{n-1}}\}$ . Hence  $\alpha^{q^n} = \alpha^{q^j}$  for some  $j$  with  $0 \leq j \leq n-1$ . Hence  $k|n-j$ , particularly  $k \leq n$ . Let  $n = uk + r$  with  $0 \leq r < k$ . Suppose that  $r > 0$ . Then each of the  $r$  elements  $\{\alpha, \alpha^q, \dots, \alpha^{q^{r-1}}\}$  appears  $u+1$  times in the total set  $\{\alpha, \alpha^q, \dots, \alpha^{q^{n-1}}\}$ , and each of the  $k-r$  elements  $\{\alpha^{q^r}, \dots, \alpha^{q^{k-1}}\}$   $u$ -times. On the other hand, since  $\beta = \alpha^{q^i}$ , each of the  $r$  elements  $\{\alpha^{q^i}, \alpha^{q^{i+1}}, \dots, \alpha^{q^{i+r-1}}\}$  appears  $u+1$  times in the total set, and each of the  $k-r$  elements  $\{\alpha^{q^{i+r}}, \dots, \alpha^{q^{i+k-1}}\}$   $u$ -times. Therefore both  $\alpha^{q^{r-1}}$  and  $\alpha^{q^{i+r-1}}$  appear  $u+1$  times in the total set,

but their  $q$ -th power  $u$  times. Hence those elements must coincide. Hence  $\alpha = \alpha^{q^i} = \beta$ , which is a contradiction.  $\square$

*Continuation of the proof of Theorem 6.4.* (iii) It is obvious that  $\Phi_n(P_\alpha) \in B(\mathbb{F}_q)$  for any  $\alpha \in \mathbb{F}_{q^n}$ . Conversely, if  $\Phi_n(P_\alpha) \in B(\mathbb{F}_q)$ , then  $\varphi_n(\alpha) = \alpha^{1+q+\dots+q^{n-1}} \in \mathbb{F}_q$ . Hence  $1 = (\alpha^{1+q+\dots+q^{n-1}})^{q-1} = \alpha^{q^n-1}$  if  $\alpha \neq 0$ , which means  $\alpha \in \mathbb{F}_{q^n}$ .  $\square$

From Theorem 6.4, we know the number  $N_q(B_n)$  of  $\mathbb{F}_q$ -points of  $B_n$ .

**Theorem 6.6**

$$N_q(B_n) = \frac{1}{n}q^n + \sum_{\substack{d|n \\ \text{with} \\ d \neq n}} \frac{1}{d} \prod_{\substack{l:\text{prime} \\ \text{with} \\ l|\frac{n}{d}}} \left(1 - \frac{1}{l}\right) q^d + 1.$$

*Proof.* By (ii) and (iii) of Theorem 6.4,

$$N_q(B_n) = \sum_{k|n} \frac{1}{k} \# \{P \in \mathbb{P}^1 \mid \mathbb{F}_q(P) = \mathbb{F}_{q^k}\}.$$

Since  $\mathbb{P}^1(\overline{\mathbb{F}}_q) = \overline{\mathbb{F}}_q \cup \{(1, 0)\}$ ,

$$\# \{P \in \mathbb{P}^1 \mid \mathbb{F}_q(P) = \mathbb{F}_{q^k}\} = \begin{cases} \# \{\alpha \in \mathbb{F}_{q^k} \mid \mathbb{F}_q(\alpha) = \mathbb{F}_{q^k}\} & \text{if } k > 1 \\ \# \mathbb{F}_q + 1 & \text{if } k = 1. \end{cases}$$

Now we compute  $\# \{\alpha \in \mathbb{F}_{q^k} \mid \mathbb{F}_q(\alpha) = \mathbb{F}_{q^k}\}$  exactly. Let  $k = l_{k,1}^{e_1} \dots l_{k,r_k}^{e_{r_k}}$  be the decomposition of  $k$  as a product of powers of distinct prime numbers. Here we understand the right hand of the decomposition of 1 to be empty. Since

$$\{\alpha \in \mathbb{F}_{q^k} \mid \mathbb{F}_q(\alpha) = \mathbb{F}_{q^k}\} = \mathbb{F}_{q^k} \setminus \bigcup_{i=1}^{r_k} \mathbb{F}_{q^{k/l_{k,i}}},$$

the cardinality of this set is

$$q^k - \sum_{i=1}^{r_k} q^{k/l_{k,i}} + \sum_{i < j} q^{k/l_{k,i}l_{k,j}} - \dots = \sum_{s=0}^{r_k} (-1)^s \sum_{i_1 < \dots < i_s} q^{k/l_{k,i_1} \dots l_{k,i_s}}.$$

When  $k = 1$ , we understand the above sum to be just  $q$ . Hence

$$N_q(B_n) = \sum_{k|n} \frac{1}{k} \sum_{s=0}^{r_k} (-1)^s \sum_{i_1 < \dots < i_s} q^{k/l_{k,i_1} \dots l_{k,i_s}} + 1.$$

For each  $d$  with  $d|n$ , we gather together the terms in  $q^d$ . If  $d = n$ , the only term is  $\frac{1}{n}q^n$ . If  $d < n$ , put  $\frac{n}{d} = m_1^{f_1} \dots m_u^{f_u}$ , where  $m_1, \dots, m_u$  are distinct prime numbers with  $f_1, \dots, f_u \geq 1$ . It is obvious that for a fixed  $d$ ,

$$\{k \mid k/l_{k,j_1} \dots l_{k,j_s} = d \text{ for some } j_1 < \dots < j_s\} = \{m_{i_1} \dots m_{i_s} d \mid i_1 < \dots < i_s\}.$$

Hence the coefficient of  $q^d$  is

$$\sum_{s=0}^u \sum_{i_1 < \dots < i_s} (-1)^s \frac{1}{m_{i_1} \cdots m_{i_s} d} = \frac{1}{d} \prod_{i=1}^u \left(1 - \frac{1}{m_i}\right) = \frac{1}{d} \prod_{\substack{l \mid \frac{n}{d} \\ l: \text{ prime}}} \left(1 - \frac{1}{l}\right).$$

This completes the proof.

**Remark 6.7** The order-sequence, the  $q$ -Frobenius index and the  $q$ -Frobenius order-sequence for  $\Phi_n : \mathbb{P}^1 \rightarrow B_n$  can be also calculated. Namely, the order-sequence for  $\Phi_n$  is  $0 < 1 < q < q^2 < \dots < q^{n-1}$ , the  $q$ -Frobenius index is 1, and hence the  $q$ -Frobenius order-sequence is  $0 < q < q^2 < \dots < q^{n-1}$ . At present, our proof of this fact is not concise. So we will discuss it in a separate paper.

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